AIAA JOURNAL
Vol. 46, No. 12, December 2008

Technical Notes

TECHNICAL NOTES are short manuscripts describing new developments or important results of a preliminary nature. These Notes should not exceed 2500 words (where a figure or table counts as 200 words). Following informal review by the Editors, they may be published within a few months of the date of receipt. Style requirements are the same as for regular contributions (see inside back cover).

Buckling of Rectangular Plates on an Elastic Foundation Using the Levy Method

L. H. Yu*
National Chung Cheng University, Chia-Yi 621, Taiwan, Republic of China
and
C. Y. Wang†
Michigan State University, East Lansing, Michigan 48824

DOI: 10.2514/1.37166

Nomenclature

b	=	half aspect ratio
c_1, c_2	=	constants
D	=	flexural rigidity
f	=	function of y
h	=	N_1L^2/D
i	=	$\sqrt{-1}$
k	=	N_2L^2/D
L	=	width
m, n	=	integers
N_1	=	compression force per length in the x direction
N_2	=	compression force per length in the y direction
s, t	=	real constants
u, υ	=	functions of y
w	=	lateral deflection
x, y	=	Cartesian coordinates
α	=	$n\pi$
$\beta, \bar{\beta}$	=	constants in the exponent
γ	=	constant defined by Eq. (21) or Eq. (22)
Δ	=	discriminant
Λ	=	spring constant of the foundation
λ	=	$L(\Lambda/D)^{1/4}$
ν	=	Poisson's ratio
$'$	=	dimensional

Introduction

THE buckling of plates is important in load-bearing panels, especially for aircraft. The analysis of the stability of thin plates can be found in [1,2]. There are cases, however, when the plate is supported by an elastic foundation that affects the buckling load. The buckling of a long rectangular plate on an elastic foundation was discussed by Seide [3]. The corresponding square plate compressed on all sides was solved by Warren [4]. For two opposite simply supported edges, the Levy analytical method can be used; otherwise, numerical methods are necessary (e.g., [5]). There are some papers on the axisymmetric buckling of circular plates [6–9], but Wang [10] recently showed that for a plate on a foundation, the buckling mode cannot be assumed a priori.

The purpose of this Note is to study the buckling of a rectangular plate on an elastic foundation. Two opposite sides are simply supported, and the Levy method is used. Our exact results will be a benchmark for numerical computations on more complex problems.

Formulation

Figure 1 shows a plate with width L and height $2bL$ subjected to constant biaxial compressive forces per length, N_1 and N_2, in the Cartesian x' and y' directions, respectively. The Kirchkoff plate equation with elastic support is

$$D\nabla^4 w + N_1 w_{x'x'} + N_2 w_{y'y'} + \Lambda w = 0 \tag{1}$$

Normalize lengths by L. Equation (1) becomes

$$\nabla^4 w + h w_{xx} + k w_{yy} + \lambda^4 w = 0 \tag{2}$$

where

$$h = \frac{N_1 L^2}{D}, \qquad k = \frac{N_2 L^2}{D}, \qquad \lambda^4 = \frac{\Lambda L^4}{D} \tag{3}$$

are important nondimensional parameters representing the buckling forces and the foundation elasticity. If the two edges parallel to the y axis are simply supported, as shown in Fig. 1, the Levy method assumes

$$w = \sin(\alpha x) f(y) \tag{4}$$

Then Eq. (2) becomes

$$f'''' + (k - 2\alpha^2) f'' + (\alpha^4 - h\alpha^2 + \lambda^4) f = 0 \tag{5}$$

Depending on the relative magnitudes of k, h, α, and λ, one can find four independent solutions. The problem can be greatly simplified if we consider symmetric loading conditions on the top and bottom edges. Then it is possible to separate the odd and even solutions of Eq. (5), corresponding to the odd and even buckling modes. Suppose the two odd or the two even solutions are represented by

$$f = c_1 u(y) + c_2 \upsilon(y) \tag{6}$$

Then only the $y = b$ boundary condition needs to be satisfied. Thus, if the edge conditions on $y = \pm b$ are clamped, we have

$$f(b) = 0, \qquad f'(b) = 0 \tag{7}$$

if they are simply supported,

$$f(b) = 0, \qquad f''(b) = 0 \tag{8}$$

and if they are free [2],

$$f''(b) - \nu\alpha^2 f(b) = 0, \qquad f'''(b) - [\alpha^2(2-\nu) - k] f'(b) = 0 \tag{9}$$

For nontrivial c_1 or c_2, any set of Eqs. (7–9) would give a

Received 15 February 2008; revision received 1 June 2008; accepted for publication 19 June 2008.

*Assistant Professor, Department of Mathematics.
†Professor, Departments of Mathematics and Mechanical Engineering.

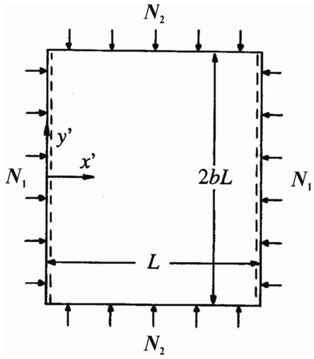

Fig. 1 The rectangular plate with vertical edges simply supported.

characteristic equation ($\nu = 0.3$ in our computations). The buckling load h or k can then be found by bisection to any accuracy. In what follows, we shall study three important loading conditions.

Uniaxial Load in the *x* Direction

In this case, N_2 or k is zero. Let $f = \exp(\beta y)$ and Eq. (5) gives the indicial equation:

$$\beta^4 - 2\alpha^2\beta^2 + (\alpha^4 - h\alpha^2 + \lambda^4) = 0 \tag{10}$$

Solving

$$\beta^2 = \alpha^2 \pm \sqrt{h\alpha^2 - \lambda^4} \tag{11}$$

If $\Delta = h\alpha^2 - \lambda^4 > 0$, then β^2 is real. If, further, $\alpha^2 > \sqrt{\Delta}$, then β^2 is positive. Let

$$\beta_1 = \sqrt{\alpha^2 + \sqrt{\Delta}}, \qquad \beta_2 = \sqrt{\alpha^2 - \sqrt{\Delta}} \tag{12}$$

Then the even solution is

$$f = c_1 \cosh(\beta_1 y) + c_2 \cosh(\beta_2 y) \tag{13}$$

and the odd solution is

$$f = c_1 \sinh(\beta_1 y) + c_2 \sinh(\beta_2 y) \tag{14}$$

If $\alpha^2 < \sqrt{\Delta}$, let $\bar{\beta}_2 = \sqrt{\sqrt{\Delta} - \alpha^2}$. The even and odd solutions are

$$f = c_1 \cosh(\beta_1 y) + c_2 \cos(\bar{\beta}_2 y) \tag{15}$$

$$f = c_1 \sinh(\beta_1 y) + c_2 \sin(\bar{\beta}_2 y) \tag{16}$$

If $\Delta = h\alpha^2 - \lambda^4 < 0$,

$$\beta^2 = \alpha^2 \pm \sqrt{\lambda^4 - h\alpha^2}\, i \tag{17}$$

where $i = \sqrt{-1}$. Let

$$\beta_1 = \sqrt{\alpha^2 + \sqrt{\lambda^4 - h\alpha^2}\, i} = s + ti, \qquad \beta_2 = \sqrt{\alpha^2 - \sqrt{\lambda^4 - h\alpha^2}\, i} = s - ti \tag{18}$$

The solutions are

$$f = c_1 \cosh(sy)\cos(ty) + c_2 \sinh(sy)\sin(ty) \tag{19}$$

$$f = c_1 \sinh(sy)\cos(ty) + c_2 \cosh(sy)\sin(ty) \tag{20}$$

The boundary conditions Eqs. (7–9) then yield exact buckling load h, depending on the preceding scenarios.

First consider the plate being simply supported on all sides, for which an exact solution is possible. The boundary-condition equation (8) dictates that f should be one of the following forms:

$$f = c_1 \cos(\gamma y), \qquad \gamma = \left(m - \frac{1}{2}\right)\frac{\pi}{b} \tag{21}$$

or

$$f = c_1 \sin(\gamma y), \qquad \gamma = m\frac{\pi}{b} \tag{22}$$

Equation (5) then yields

$$h = \frac{(\gamma^2 + \alpha^2)^2 + \lambda^4}{\alpha^2} \tag{23}$$

The minimum h occurs at minimum γ or $m = 1$ in Eq. (21) and f is even. Equation (23) gives

$$h = \frac{[\pi/(2b)]^4 + \lambda^4}{(n\pi)^2} + 2\left(\frac{\pi}{2b}\right)^2 + (n\pi)^2 \tag{24}$$

Table 1 shows the results for minimum h.

If the edges are clamped at $y = \pm b$, the solution is not in closed form. In the range studied, we find that minimum h occurs when $\alpha^2 < \sqrt{\Delta}$ and the function f is always even. Equations (7) and (15) give the characteristic equation:

$$\beta_1 \sinh(\beta_1 b)\cos(\bar{\beta}_2 b) + \bar{\beta}_2 \sin(\bar{\beta}_2 b)\cosh(\beta_1 b) = 0 \tag{25}$$

The results are given in Table 2.

There are some previous published works on the horizontally clamped case when the foundation is absent ($\lambda = 0$). Timoshenko

Table 1 Buckling load *h* for uniaxial loads in the *x* direction with simply supported edges[a]

blambda	0	2	5	10	20
0.1	986.97 (5)	987.03 (5)	989.48 (5)	1207.5 (5)	1433.8 (7)
0.25	157.90 (2)	158.33 (2)	173.74 (2)	297.91 (3)	888.99 (6)
0.5	39.23 (1)	41.10 (1)	77.51 (2)	222.25 (3)	825.64 (6)
1	15.42 (1)	17.04 (1)	60.40 (2)	206.41 (3)	810.60 (6)
5	10.07 (1)	11.69 (1)	55.50 (2)	201.61 (3)	805.82 (6)

[a]The number in parenthesis denotes the number of waves n in the x direction.

Table 2 Buckling load *h* for uniaxial loads in the *x* direction with clamped parallel edges[a]

blambda	0	2	5	10	20
0.1	1726.9 (8)	1727.0 (8)	1727.9 (8)	1742.7 (8)	1980.3 (8)
0.25	275.23 (3)	275.39 (3)	282.27 (3)	366.95 (4)	920.82 (6)
0.5	75.92 (2)	76.32 (2)	91.74 (2)	230.22 (3)	829.04 (6)
1	18.97 (1)	20.60 (1)	61.78 (2)	207.24 (3)	811.00 (6)
5	10.09 (1)	11.71 (1)	55.52 (2)	201.61 (3)	805.82 (6)

[a]The number in parenthesis denotes the number of waves n in the x direction.

and Gere [11] published results for aspect ratios $(1/2b)$ from 0.4 to 1. Because of different normalization, their values are related to ours by multiplying by the aspect ratio and π. Thus, for $b = 0.5$ (square plate) they obtained 75.90, whereas we have 75.92. For $b = 1$ the values are identical at 18.97. Our values are identical to those in [2]: that is, for $b = 0.5$ the buckling force is 75.92, whereas for $b = 0.25$ it is 275.23.

If the edges are free at $y = \pm b$, we find that minimum h occurs when $\alpha^2 > \sqrt{\Delta}$ and the function f is always even. Equations (9) and (13) give the characteristic equation:

$$\beta_2(\beta_1^2 - \nu\alpha^2)[\beta_2^2 - \alpha^2(2-\nu)]\cosh(\beta_1 b)\sinh(\beta_2 b) - \beta_1(\beta_2^2 - \nu\alpha^2)[\beta_1^2 - \alpha^2(2-\nu)]\sinh(\beta_1 b)\cosh(\beta_2 b) = 0 \tag{26}$$

The results are given in Table 3.

Uniaxial Load in the *y* Direction

In this case, N_1 or h is zero. Let $f = \exp(\beta y)$ and Eq. (5) gives the indicial equation:

$$\beta^4 - (2\alpha^2 - k)\beta^2 + (\alpha^4 + \lambda^4) = 0 \tag{27}$$

Solving

$$\beta^2 = \frac{2\alpha^2 - k \pm \sqrt{\Delta}}{2} \tag{28}$$

where

$$\Delta = k^2 - 4\alpha^2 k - 4\lambda^4 \tag{29}$$

We can show that $\Delta < 0$ if

$$k < 2\alpha^2 + 2\sqrt{\alpha^4 + \lambda^4} \tag{30}$$

and $\Delta > 0$ otherwise. If $\Delta > 0$ and $2\alpha^2 - k > \sqrt{\Delta}$, then the solutions are Eqs. (13) and (14), in which

$$\beta_1 = \sqrt{\frac{2\alpha^2 - k + \sqrt{\Delta}}{2}}, \qquad \beta_2 = \sqrt{\frac{2\alpha^2 - k - \sqrt{\Delta}}{2}} \tag{31}$$

If $\Delta > 0$ and $2\alpha^2 - k < \sqrt{\Delta}$, then the solutions are Eqs. (15) and (16), in which β_1 as in Eq. (31) and

$$\bar{\beta}_2 = \sqrt{\frac{\sqrt{\Delta} - 2\alpha^2 + k}{2}} \tag{32}$$

If $\Delta < 0$, let

$$\sqrt{\frac{2\alpha^2 - k + \sqrt{4\lambda^4 + 4\alpha^2 k - k^2}\,i}{2}} = s + ti \tag{33}$$

and the solutions are in the form of Eqs. (19) and (20).

For the case when all edges are simply supported, tilt the plate 90 deg so that the forces in the *y* direction become the forces in the *x* direction. Then the solution is the same as that in the previous section. Note, however, that the normalization length is different.

If the edges at $y = \pm b$ are clamped, we find that minimum h occurs when $2\alpha^2 - k < \sqrt{\Delta}$, f is even, and $n = 1$. Equation (25) is used, but with β_1 and $\bar{\beta}_2$ defined as in Eqs. (31) and (32). Table 4 shows the results.

If the edges at $y = \pm b$ are free, we find that minimum h occurs when $k^2 - 4\alpha^2 k - 4\lambda^4 < 0$, f could be even or odd, and $n = 1$. Either Eq. (19) or Eq. (20) is used, but with s and t defined as in Eq. (33). The boundary-condition equation (9) then gives a more complicated characteristic equation. Table 5 shows the results.

Table 3 Buckling load *h* for uniaxial loads in the *x* direction with free parallel edges[a]

blambda	0	2	5	10	20
0.1	9.018 (1)	10.64 (1)	52.29 (2)	195.58 (3)	791.41 (6)
0.25	9.169 (1)	10.79 (1)	53.42 (2)	198.34 (3)	798.97 (6)
0.5	9.400 (1)	11.02 (1)	54.26 (2)	199.74 (3)	801.91 (6)
1	9.604 (1)	11.23 (1)	54.73 (2)	200.48 (3)	803.44 (6)
5	9.803 (1)	11.42 (1)	55.13 (2)	201.04 (3)	804.29 (6)

[a]The number in parenthesis denotes the number of waves *n* in the *x* direction.

Table 4 Buckling load *k* for uniaxial loads in the *y* direction applied to clamped edges ($n = 1$ in all cases)

blambda	0	2	5	10	20
0.1	1006.98	1007.05	1008.89	1037.36	1481.02
0.25	179.51	179.80	191.32	355.55	955.12
0.5	66.55	67.75	111.28	254.43	547.00
1	52.06	54.02	81.76	231.62	830.19
5	39.88	41.46	73.91	221.12	820.36

Table 5 Buckling load *k* for uniaxial loads in the *y* direction applied to free edges ($n = 1$ in all cases)

blambda	0	2	5	10	20
0.1	14.12[a]	14.17[a]	16.18[a]	47.09[a]	382.51
0.25	15.62[a]	15.94[a]	28.03[a]	100.46	409.70[a]
0.5	20.16[a]	21.23[a]	37.26	113.21[a]	413.88
1	22.32	23.18	40.21[a]	114.21	413.92
5	22.80	23.65	40.37	114.21	413.92

[a]The mode is odd in *y*.

Equal Biaxial Load

We set $h = k$ in this case. Using $f = \exp(\beta y)$, Eq. (5) yields

$$\beta^4 + (k - 2\alpha^2)\beta^2 + (\alpha^4 - k\alpha^2 + \lambda^4) = 0 \tag{34}$$

The solution is

$$\beta^2 = \frac{2\alpha^2 - k \pm \sqrt{\Delta}}{2} \tag{35}$$

where Δ is simple:

$$\Delta = k^2 - 4\lambda^4 \tag{36}$$

Thus, if $\Delta > 0$ and $2\alpha^2 - k > \sqrt{\Delta}$, the solutions are Eqs. (13) and (14), in which β_1 and β_2 have the same form as in Eq. (31). If $\Delta > 0$

Table 6 Buckling load *k* for equal biaxial loads with simply supported edges[a]

blambda	0	2	5	10	20
0.1	256.61 (1,1)	256.67 (1,1)	259.05 (1,1)	295.56 (1,1)	800.04 (1,4)
0.25	49.35 (1,1)	49.67 (1,1)	62.02 (1,1)	205.61 (1,2)	800.04 (1,5)[b]
0.5	19.74 (1,1)	20.55 (1,1)	51.41 (1,1)	200.02 (1,3) (2,1)	800.04 (3,4) (2,5)[b]
1	12.33 (1,1)	13.63 (1,1)	51.41 (1,1)[b]	200.02 (3,2)	800.04 (4,5) (5,4)[b]
5	9.967 (1,1)	11.57 (1,1)	50.04 (1,6)[b]	200.00 (6,3)	799.98 (11,6)

[a]Parentheses denote modes (m, n).
[b]The mode is odd in *y*.

Table 7 Buckling load k for equal biaxial loads with two clamped edges and two simply supported edges[a]

blambda	0	2	5	10	20
0.1	927.99 (4)	927.45 (4)	928.73 (4)	948.39 (4)	1228.01 (5)
0.25	150.99 (1)	151.24 (1)	158.13 (2)	250.65 (3)	813.62 (6)
0.5	37.80 (1)	38.51 (1)	65.19 (1)[c]	203.40 (3)	803.44 (6)[c]
1	14.62 (1)	15.86 (1)	54.58 (1)[bc]	200.87 (3)[c]	801.00 (6)[bc]
5	9.979 (1)	11.59 (1)	50.23 (1)[bc]	200.05 (3)[bc]	800.04 (6)[c]

[a]Parentheses denote the number n.
[b]The mode is odd in y.
[c]The form is either Eq. (13) or Eq. (14); otherwise, the form is Eq. (15).

Table 8 Buckling load k for equal biaxial loads, with two free edges and two simply supported edges

blambda	0	2	5	10	20
0.1	8.994	10.61	15.67[ab]	45.62[ab]	379.74[b]
0.25	9.060	10.65	23.48[ab]	96.28[b]	404.37[ab]
0.5	9.199	10.76	31.80[b]	107.50[ab]	408.81[b]
1	9.400	10.94	33.03[ab]	108.72[b]	408.85[b]
5	9.629	11.12	33.24[b]	108.72[b]	408.85[b]

[a]The mode is odd in y.
[b]The form is either Eq. (19) or Eq. (20); otherwise, the form is Eq. (13).

and $2\alpha^2 - k < \sqrt{\Delta}$, then the solutions are Eqs. (15) and (16), in which $\bar{\beta}_2$ has the same form as Eq. (32). If $\Delta < 0$, let

$$\sqrt{\frac{2\alpha^2 - k + \sqrt{4\lambda^4 - k^2}i}{2}} = s + ti \tag{37}$$

and the solutions are from Eqs. (19) and (20).

If all edges are simply supported, one can assume Eqs. (21) and (22). Then Eq. (5) yields the exact solution:

$$k = \frac{(\gamma^2 + \alpha^2)^2 + \lambda^4}{\gamma^2 + \alpha^2} \tag{38}$$

In this case, there are nodal lines in both directions. Results are shown in Table 6.

Warren [4] considered the simply supported square plate under equal biaxial load, but gave no numerical values to compare with our results.

For clamped horizontal edges, not only do we encounter even and odd modes, but also a switch in the mode form. Table 7 shows the results. For free horizontal edges, we find that n is always 1, but there are switches in mode form, as shown in Table 8.

Conclusions

The present Note considers the fundamental buckling problem of a rectangular plate on an elastic foundation. The results are reported here for the first time. Our tables are also exact and can serve as benchmarks for numerical computations.

The separation of even and odd modes greatly facilitates the comprehensive analyses. Depending on the stiffness of the elastic foundation, there are three different functional forms. All three forms are used in this study. Because of limited space, we have omitted the rare equality cases in the inequality conditions.

The following general properties can be concluded (some are obvious, some are not so obvious): The buckling load increases with the stiffness of the foundation. The buckling load is lowest for the free edge and highest for the clamped edge and less if biaxially loaded. The number of waves of the buckling mode increases with stiffness and occurs in the direction of the applied stress. An increase in b or the height increases the buckling load for the free horizontal edge case, but the effect is the opposite for the clamped or simply supported cases. This is because the end conditions for a plate strip (large b) are more restrictive than for a free edge, but less restrictive than for the simply supported or clamped cases. As b increases indefinitely, the buckling load tends to a constant. This is especially true for large λ, for which the effect of the foundation stiffness dampens the edge effects.

Specifically, if the load is perpendicular to the simply supported edges (uniaxial load in the x direction), the function $f(y)$ is always even. If the load is parallel to the simply supported edges (uniaxial load in the y direction), the function $f(y)$ may be even or odd, but there is only one half-wave in x ($n = 1$). If the plate is biaxially loaded, buckled waves can occur in both directions. The biaxial buckling force is about half of that of the uniaxial loads when stiffness and b are small, but the difference is less when stiffness or b are large.

The Levy method may also be used in other cases in which the horizontal edges have different combinations of constraints. In these asymmetric cases the function $f(y)$ cannot be separated into even and odd forms as in the present Note, and the buckling characteristics will be difficult, if not impossible, to analyze.

References

[1] Timoshenko, S. P., and Gere, J. M., *Theory of Elastic Stability*, 2nd ed., McGraw–Hill, New York, 1961, Chap. 7.
[2] Wang, C. M., Wang, C. Y., and Reddy, J. N., *Exact Solutions for Buckling of Structural Members*, CRC Press, Boca Raton, FL, 2005, Chap. 4.
[3] Seide, P., "Compressive Buckling of a Long Simply Supported Plate on an Elastic Foundation," *Journal of Aerospace Science*, Vol. 25, No. 6, 1958, pp. 382–394.
[4] Warren, R. C., "Buckling of a Rectangular Plate on an Elastic Foundation, Compressed in Two Directions," Ph.D. Thesis, Michigan State Univ., East Lansing, MI, 1980.
[5] Raju, K. K., and Rao, G. V., "Thermal Post-Buckling of a Square Plate Resting on an Elastic Foundation by Finite Element Method," *Computers and Structures*, Vol. 28, No. 2, 1988, pp. 195–199. doi:10.1016/0045-7949(88)90039-9
[6] Yu, Y. Y., "Axisymmetrical Bending of Circular Plates Under Simultaneous Action of Lateral Load, Force in the Middle Plane, and Elastic Foundation," *Journal of Applied Mechanics*, Vol. 24, Mar. 1957, pp. 141–143.
[7] Galletly, G. D., "Circular Plates on a Generalized Elastic Foundation," *Journal of Applied Mechanics*, Vol. 26, June 1959, p. 297.
[8] Kline, L. V., and Hancock, J. O., "Buckling of Circular Plate on Elastic Foundation," *Journal of Engineering for Industry*, Vol. 87, Aug. 1965, pp. 323–324.
[9] Wolkowisky, J. H., "Buckling of the Circular Plate Embedded in Elastic Springs: An Application to Geophysics," *Communications on Pure and Applied Mathematics*, Vol. 22, No. 5, 1969, pp. 639–667. doi:10.1002/cpa.3160220506
[10] Wang, C. Y., "On the Buckling of a Circular Plate on an Elastic Foundation," *Journal of Applied Mechanics*, Vol. 72, No. 5, 2005, pp. 795–796. doi:10.1115/1.1988347
[11] Timoshenko, S. P., and Gere, J. M., *Theory of Elastic Stability*, 2nd ed., McGraw–Hill, New York, 1961.

F. Pai
Associate Editor